Derivative of the Log Determinant

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1 Introduction

Inspired by a stack exchange article that I was too lazy to recover, I sought to show the following:

Suppose $C(\alpha)$ is a Hermitian positive-definite matrix for every value of α and that the parameter α is such that the eigenvalues and eigenvectors may be thought of as smooth functions of α i.e. their first partial derivative with respect to α exists everywhere that is relevant to the problem. Then

$$\frac{\partial \ln |C|}{\partial \alpha} = \mathrm{tr}\left(C^{-1}\frac{\partial C}{\partial \alpha}\right) \tag{1}$$

2 Derivation

For any value of α, we may write

$$\ln |C(\alpha)| = \sum_i \ln \lambda_i(\alpha) \tag{2}$$

where the λ_i are the eigenvalues of C. Then clearly

$$\frac{\partial \ln |C(\alpha)|}{\partial \alpha} = \sum_i \frac{1}{\lambda_i(\alpha)}\frac{\partial \lambda_i}{\partial \alpha}. \tag{3}$$

Note that we may write C as

$$\sum_i \lambda_i(\alpha) v_i(\alpha) v_i^\dagger(\alpha) \tag{4}$$

where $v_i(\alpha)$ are the eigenvectors of $C(\alpha)$. From now on I will omit the argument and it should just be understood that everything here is implicitly a function of α. Similarly, we may write

$$C^{-1} = \sum_i \frac{1}{\lambda_i} v_i v_i^\dagger. \tag{5}$$

Taking the derivative of C with respect to α, we have

$$\frac{\partial C}{\partial \alpha} = \sum_i \frac{\partial \lambda_i}{\partial \alpha} v_i v_i^\dagger + \lambda_i \left(\frac{\partial v_i}{\partial \alpha} v_i^\dagger + v_i \frac{\partial v_i^\dagger}{\partial \alpha} \right) \tag{6}$$

Now note that

$$\mathrm{tr}\left(C^{-1} \frac{\partial C}{\partial \alpha} \right) = \sum_i v_i^\dagger C^{-1} \frac{\partial C}{\partial \alpha} v_i. \tag{7}$$

Meanwhile, since

$$v_i^\dagger v_j = \delta_{ij}, \tag{8}$$

we have

$$v_i^\dagger \frac{\partial v_j}{\partial \alpha} = -\frac{\partial v_i^\dagger}{\partial \alpha} v_j. \tag{9}$$

From equations 5, we have

$$v_i^\dagger C^{-1} = \frac{1}{\lambda_i} v_i^\dagger. \tag{10}$$

By combining Equations 6 and 9 we get

$$\frac{\partial C}{\partial \alpha} v_i = \frac{\partial \lambda_i}{\partial \alpha} v_i + \lambda_i \frac{\partial v_i}{\partial \alpha} + \sum_j \lambda_j v_j \frac{\partial v_j^\dagger}{\partial \alpha} v_i \tag{11}$$

$$= \frac{\partial \lambda_i}{\partial \alpha} v_i + \lambda_i \frac{\partial v_i}{\partial \alpha} - \sum_j \lambda_j v_j v_j^\dagger \frac{\partial v_i}{\partial \alpha} \tag{12}$$

Finally, combining Equations 3, 10, and 12, we have

$$\mathrm{tr}\left(C^{-1} \frac{\partial C}{\partial \alpha} \right) = \sum_i \frac{1}{\lambda_i} \frac{\partial \lambda_i}{\partial \alpha} + \cancel{\lambda_i \left(v_i^\dagger \frac{\partial v_i}{\partial \alpha} - v_i^\dagger \frac{\partial v_i}{\partial \alpha} \right)} = \frac{\partial \ln |C(\alpha)|}{\partial \alpha} \tag{13}$$