Real and Imaginary Covariances

Mike Wilensky

October 2021

1 Introduction

This is a short note on how to get to the covariance matrix for a real-valued multivariate normal distribution that was constructed from an initially complex one. We start with

$$\mathbf{C} = \langle \mathbf{z}\mathbf{z}^\dagger \rangle \tag{1}$$

$$\mathbf{\Gamma} = \langle \mathbf{z}\mathbf{z}^T \rangle \tag{2}$$

where I've implicitly assumed mean 0 or equivalently that the mean is subtracted off in the definition of $\mathbf{z}$. This is just to make the notation a little less messy, but the final answer doesn't depend on them so I've decided to omit them. These two matrices (along with the implicit mean), define a multivariate complex gaussian random variable.

To calculate the covariance matrix of its individual components, we form

$$\mathbf{C}_\Re = \frac{1}{4}\langle (\mathbf{z}+\mathbf{z}^*)(\mathbf{z}+\mathbf{z}^*)^T \rangle = \frac{1}{2}\Re\Big[\mathbf{C}+\mathbf{\Gamma}\Big] \tag{3}$$

$$\mathbf{C}_\Im = -\frac{1}{4}\langle (\mathbf{z}-\mathbf{z}^*)(\mathbf{z}-\mathbf{z}^*)^T \rangle = \frac{1}{2}\Re\Big[\mathbf{C}-\mathbf{\Gamma}\Big] \tag{4}$$

$$\mathbf{C}_{\Re,\Im} = \frac{1}{4i}\langle (\mathbf{z}+\mathbf{z}^*)(\mathbf{z}-\mathbf{z}^*)^T \rangle = \frac{1}{2}\Im\Big[\mathbf{\Gamma}-\mathbf{C}\Big] \tag{5}$$

$$\mathbf{C}_{\Im,\Re} = \frac{1}{4i}\langle (\mathbf{z}-\mathbf{z}^*)(\mathbf{z}+\mathbf{z}^*)^T \rangle = \frac{1}{2}\Im\Big[\mathbf{\Gamma}+\mathbf{C}\Big] \tag{6}$$

These then form a block covariance matrix, $\mathbf{K}$, for the real-valued vector, $\mathbf{x}$, as follows

$$\mathbf{x} = \begin{pmatrix} \Re(\mathbf{z}) \\ \Im(\mathbf{z}) \end{pmatrix} \tag{7}$$

$$\mathbf{K} = \begin{pmatrix} \mathbf{C}_\Re & \mathbf{C}_{\Re,\Im} \\ \mathbf{C}_{\Im,\Re} & \mathbf{C}_\Im \end{pmatrix} \tag{8}$$