# Eigenvalues of the "Realification" of a Complex Matrix

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## 1 Introduction

The point of this note is to relate the eigenspectrum of the "realification" of a complex matrix, $\mathbf{A}$, to the eigenspectrum of $\mathbf{A}$. By realification, I mean that if

$$\mathbf{A} = \Re(\mathbf{A}) + i\Im(\mathbf{A}) \tag{1}$$

then the realified form is the block matrix

$$\mathbf{A}' = \begin{pmatrix} \Re(\mathbf{A}) & -\Im(\mathbf{A}) \\ \Im(\mathbf{A}) & \Re(\mathbf{A}) \end{pmatrix}. \tag{2}$$

This is a useful form since this preserves the complex algebra of $\mathbf{A}$ in the sense that if

$$\mathbf{z} = \Re(\mathbf{z}) + i\Im(\mathbf{z}) \tag{3}$$

and we define a block vector $\mathbf{x}$,

$$\mathbf{x} = \begin{pmatrix} \Re(\mathbf{z}) \\ \Im(\mathbf{z}) \end{pmatrix}, \tag{4}$$

then

$$\mathbf{A}'\mathbf{x} = \begin{pmatrix} \Re(\mathbf{Az}) \\ \Im(\mathbf{Az}). \end{pmatrix} \tag{5}$$

## 2 Useful Properties of Determinants

Now I go over some useful properties of determinants. First, it can be shown that for some block matrix, $\mathbf{B}$,

$$\mathbf{B} = \begin{pmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{E} \end{pmatrix} \tag{6}$$

has determinant

$$\det(\mathbf{B}) = \det(\mathbf{C})\det(\mathbf{E}). \tag{7}$$

I will also make use of the fact that the determinant of a product of matrices is equal to the product of the determinants. We also make use of the fact that

$$\det(\mathbf{M}^*) = \det(\mathbf{M}^\dagger) = \det(\mathbf{M})^* \tag{8}$$

## 3 Analyzing the Eigenspectrum

The eigenspectrum of an operator is determined by its characteristic polynomial, $P_{\mathbf{A}}(\lambda)$ which is generated by

$$P_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \prod_{j=1}^{N}(\lambda - \lambda_j). \tag{9}$$

where $\lambda_j$ is the $j$th eigenvalue. Now we show that

$$P_{\mathbf{A}'}(\lambda) = P_{\mathbf{A}}(\lambda)P_{\mathbf{A}^*}(\lambda). \tag{10}$$

$$\begin{aligned}
P_{\mathbf{A}'}(\lambda) &= \det\begin{pmatrix}\Re(\mathbf{A}) - \lambda\mathbf{I} & -\Im(\mathbf{A}) \\ \Im(\mathbf{A}) & \Re(\mathbf{A}) - \lambda\mathbf{I}\end{pmatrix} \\
&= \det\left[\begin{pmatrix}\mathbf{I} & \mathbf{0} \\ -i\mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\Re(\mathbf{A}) - \lambda\mathbf{I} & -\Im(\mathbf{A}) \\ \Im(\mathbf{A}) & \Re(\mathbf{A}) - \lambda\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{0} \\ i\mathbf{I} & \mathbf{I}\end{pmatrix}\right] \\
&= \det\begin{pmatrix}\Re(\mathbf{A}) - i\Im(\mathbf{A}) - \lambda\mathbf{I} & -\Im(\mathbf{A}) \\ \mathbf{0} & \Re(\mathbf{A}) + i\Im(\mathbf{A}) - \lambda\mathbf{I}\end{pmatrix} \\
&= \det\begin{pmatrix}\mathbf{A}^* - \lambda\mathbf{I} & -\Im(\mathbf{A}) \\ \mathbf{0} & \mathbf{A} - \lambda\mathbf{I}\end{pmatrix} \\
&= P_{\mathbf{A}}(\lambda)P_{\mathbf{A}^*}(\lambda)
\end{aligned} \tag{11}$$

where in the second equality, I have used the fact that the two matrices on either side of $\mathbf{A}'$ have unit determinant, in the third equality I have carried the matrix multiplication out, in the fourth equality I have just rewritten the line above, and in the final equality I have used the definition of the characteristic polynomial as well as the triangular block matrix determinant property from the previous section. Now note that if

$$\mathbf{A}\mathbf{z} = \lambda\mathbf{z}, \tag{12}$$

then

$$\mathbf{A}^*\mathbf{z}^* = \lambda^*\mathbf{z}^*, \tag{13}$$

which means that the eigenvectors of $\mathbf{A}^*$ are the complex conjugates of the eigenvectors of $\mathbf{A}$, with corresponding eigenvalues that are also complex conjugates of the ones for $\mathbf{A}$. Combined with the result above, this means that the $2N$ eigenvalues of $\mathbf{A}'$ are the $N$ eigenvalues of $\mathbf{A}$ and their complex conjugates.

## 4 Some Applications

Suppose $\mathbf{Z}$ is a circular complex Gaussian random vector, with covariance matrix $\mathbf{C} = \langle\mathbf{Z}\mathbf{Z}^\dagger\rangle$. Then suppose

$$\mathbf{X} = \begin{pmatrix}\Re(\mathbf{Z}) \\ \Im(\mathbf{Z})\end{pmatrix}. \tag{14}$$

It can be shown that the covariance matrix of $\mathbf{X}$ is

$$\mathbf{K} = \langle \mathbf{X}\mathbf{X}^{\mathbf{T}} \rangle = \frac{1}{2}\mathbf{C}' \tag{15}$$

where $\mathbf{C}'$ is the realification of $\mathbf{C}$. Since $\mathbf{C}$ is Hermitian, all of its eigenvalues are real. Since it is positive-definite,[1] all of these eigenvalues are positive. This means that

$$\begin{aligned}\sqrt{\det(2\pi\mathbf{K})} &= \sqrt{\det(\pi\mathbf{C}')} \\ &= \sqrt{\prod_{j=1}^{N} (\pi\lambda_j)^2} \\ &= \sqrt{\det(\pi\mathbf{C})^2} \\ &= \det(\pi\mathbf{C})\end{aligned} \tag{16}$$

which explains where the factor of 2 and the square root go in the prefactor when writing the pdf of a circular complex Gaussian random vector in complex notation as opposed to real/imaginary notation.[2]

Now suppose we are interested in calculating $\sqrt{\det(\mathbf{I} - 2it\mathbf{K}\mathbf{A}')}$ for some hermitian matrix $\mathbf{A}$ ($\mathbf{A}'$ being its realification). This comes up in the study of the thermal noise error estimates in 21-cm power spectrum measurements. Since $2\mathbf{K}\mathbf{A}'$ is the realification of $\mathbf{C}\mathbf{A}$, we have

$$\sqrt{\det(\mathbf{I} - 2it\mathbf{K}\mathbf{A}')} = \sqrt{\prod_{j=1}^{N} (1 - it\lambda_j)(1 - it\lambda_j^*)} \tag{17}$$

where $\lambda_j$ is the $j$th eigenvalue of the complex, potentially non-hermitian matrix $\mathbf{C}\mathbf{A}$. The advantage to this expression is that if all the eigenvalues are real, which I will show is true if $\mathbf{A}$ is Hermitian and invertible, *then it tells us which branch of the complex square root function to be on automatically.* This helps us calculate the characteristic function of the marginal pdf of the thermal noise fluctuations on a power spectrum measurement in terms of the covariance and some "selection matrix" $\mathbf{A}$ that tells us which bandpowers we are combining incoherently.

Now, I got this proof from stack exchange on a post by Martin Argerami.[3] Suppose $\mathbf{C}$ is hermitian and positive definite, and $\mathbf{A}$ is hermitian. Then we may write $\mathbf{C} = \mathbf{M}^\dagger\mathbf{M}$ for some matrix, $\mathbf{M}$, e.g. by Cholesky decomposition. Then we can use the fact that $\mathbf{M}^\dagger\mathbf{M}\mathbf{A}$ has the same eigenvalues as the matrix $\mathbf{M}\mathbf{A}\mathbf{M}^\dagger$, which comes from the fact that

$$P_{\mathbf{QR}}(\lambda) = P_{\mathbf{RQ}}(\lambda) \tag{18}$$

[1] If it is only positive semi-definite, then there are degenerate subspaces than can be taken out in a well-defined way to make it positive definite on the non-degenerate subspace.

[2] The factor of 2 in the exponent essentially goes away because of the factor of 2 in 15. A more detailed accounting exists in a book by Gallagher called "Stochastic Processes: Theory for Applications."

[3] https://math.stackexchange.com/questions/134884/eigenvalues-of-product-of-two-hermitian-matrices

i.e. we can swap the order of a matrix multiply and obtain the same characteristic polynomial and therefore the same eigenvalues (I believe this relies on $\mathbf{Q}$ and $\mathbf{R}$ being invertible, which we've assumed). This is clearly a Hermitian matrix and therefore has real eigenvalues, though it may not be positive-definite.