A matrix is circulant if and only if it is diagonalized by Fourier modes.

Michael Wilensky

March 2025

1 Introduction

This document is just a quick proof showing that circulant matrices are diaganalized by Fourier modes. I'm thinking of square, full rank matrices in my mind, but I don't think the details of this proof rely on those assumptions. The wikipedia article notes that this is basically just the discrete convolution theorem.

A circulant matrix is a Toeplitz matrix:

$$A_{i+1,j+1} = A_{i,j} \tag{1}$$

Furthermore and $N \times N$ circulant matrix whose top left corner is $A_{0,0}$ satisfies

$$A_{i,N-1} = A_{i+1,0} \tag{2}$$

These two properties make a matrix whose rows (or columns) "circulate," hence the name:

$$A = \begin{pmatrix} A_{0,0} & A_{0,1}, & A_{0,2}, & ..., & A_{0,N-1} \\ A_{0,N-1} & A_{0,0}, & A_{0,1}, & ..., & A_{0,N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_{0,1} & & ... & & A_{0,0} \end{pmatrix} \tag{3}$$

We do the proof in one direction, and then the other. We finish with some practical results.

2 Matrices diagonalized by Fourier modes are circulant

Suppose a matrix is diagonalized by Fourier modes:

$$A = F\Lambda F^{\dagger} \tag{4}$$

where

$$F_{j,k} = \frac{e^{2\pi i \frac{jk}{N}}}{\sqrt{N}}. \tag{5}$$

and Λ is a diagonal matrix containing the eigenvalues. This means that

$$(F^\dagger)_{j,k} = F^*_{k,j} = \frac{e^{-2\pi i \frac{jk}{N}}}{\sqrt{N}}, \tag{6}$$

which in turn means that

$$A_{j,l} = \sum_k \lambda_k \frac{e^{2\pi i \frac{(j-l)k}{N}}}{N}. \tag{7}$$

From this expression, we can see directly that $A_{j+1,l+1} = A_{j,l}$. Then, let's show some cool modular arithmetic. Note that

$$\frac{(j-(N-1))k}{N} = \frac{(j+1)k}{N} - k, \tag{8}$$

so

$$A_{j,N-1} = \sum_k \lambda_k \frac{e^{2\pi i \left(k + \frac{(j+1)k}{N}\right)}}{N}, \tag{9}$$

but

$$e^{2\pi i k} = 1 \tag{10}$$

for integer k. This leaves

$$A_{j,N-1} = A_{j+1,0}. \tag{11}$$

Both our properties have been satisfied, so we're done with this direction.

3 A circulant matrix has Fourier modes as eigenvectors

Now we go the other direction. To begin, let us define the top row of A as the (row) vector v^T. We can then write

$$A = \begin{pmatrix} v^T \\ v^T P \\ v^T P^2 \\ \vdots \\ v^T P^{N-1} \end{pmatrix} \tag{12}$$

where P is a (circulant!) matrix that circulantly shifts the elements of v^T around:

$$P = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & ... & 1 \\ 1 & 0 & 0 & ... & 0 \end{pmatrix} \tag{13}$$

Take an arbitrary column of F from above. Let's call it u_j. What is the action of P on this column? It too just circulantly shifts the column (when P acts on columns, it shifts them backwards). That is

$$(Pu_j)_k = \frac{e^{2\pi i \frac{j(k+1)}{N}}}{\sqrt{N}} = e^{2\pi i \frac{j}{N}}(u_j)_k. \tag{14}$$

Due to the same modular arithmetic we made use of above, you can apply this at any k (and j). Applying this recursively, we can see that

$$P^n u_j = e^{2\pi i \frac{jn}{N}}(u_j). \tag{15}$$

In other words u_j is an eigenvector of P^n with eigenvalue $e^{2\pi i \frac{jn}{N}}$. This means that

$$Au_j = (v^T u_j) \begin{pmatrix} 1 \\ e^{2\pi i \frac{j}{N}} \\ \vdots \\ e^{2\pi i \frac{jn}{N}} \\ \vdots \\ e^{2\pi i \frac{j(N-1)}{N}} \end{pmatrix} = (v^T u_j) u_j. \tag{16}$$

We have therefore shown that u_j (arbitrary j) is an eigenvector of arbitrary circulant A with eigenvalue $v^T u_j$.

4 Practical results

The practicality of this result lies in the fact that $v^T u_j$ is the jth discrete Fourier mode of the vector v^T. Due to the fast fourier transform (FFT), this means that circulant matrices are diagonalizable in $O(N \log N)$, by just FFTing the first row. Writing multiplication by A in terms of $FAF^\dagger$, but taking advantage of FFTs, essentially implements a fast convolution via FFT (as advertised by wikipedia). If A is full rank (no nodes in the FFT of its first row), then one can also implement a fast *deconvolution* with the kernel that A represents by instead applying A^{-1} via FFT.